Linear Algebra & Geometry

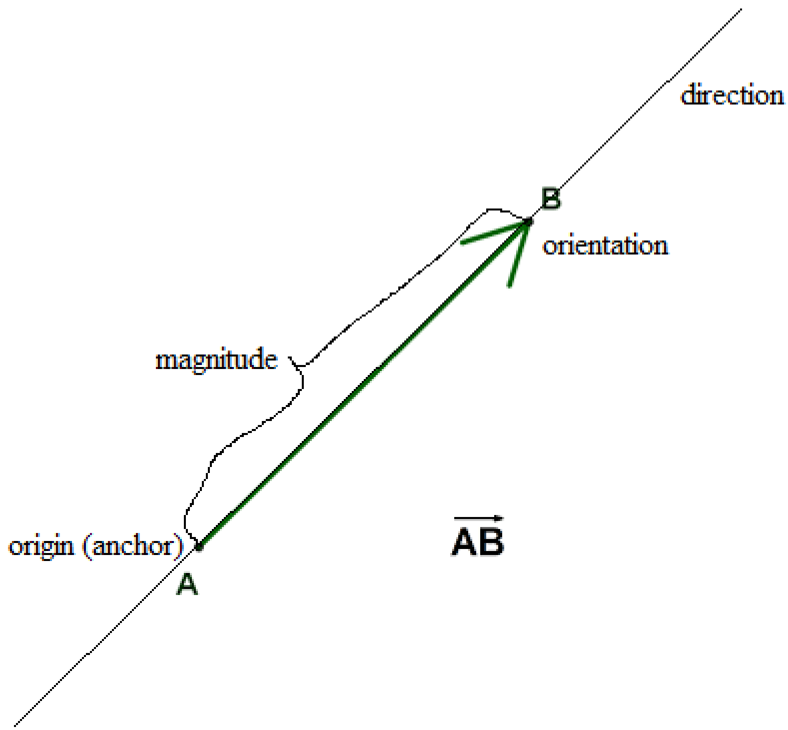
LECTURE 6

Vector spaces

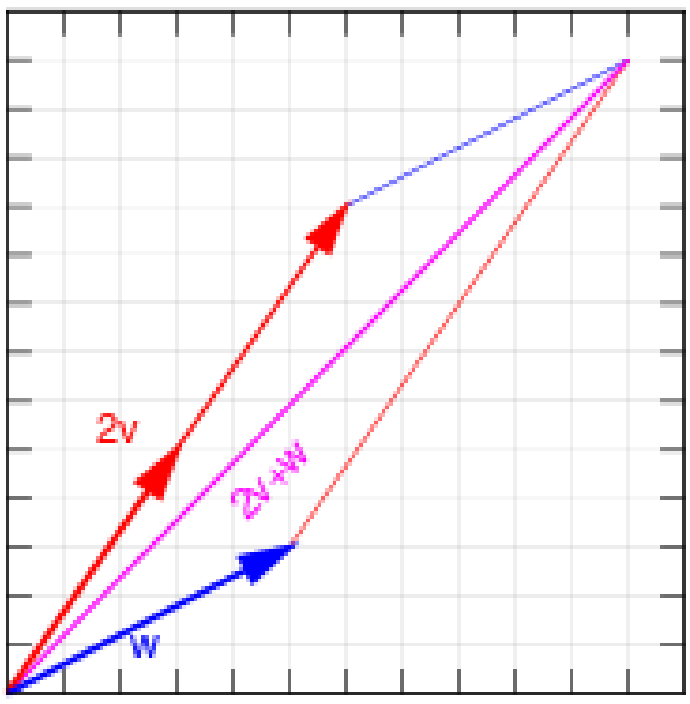
Vector Spaces

Vectors often appear in physics where they are used to represent quantities such as a force, the velocity or the acceleration of an object and others, that are not fully representable by a single number like, for example, the mass of an object or the volume of a solid or the area of a plane region. The fact that they are characterized by such properties as the magnitude, the direction, orientation and, often, a point of origin (as in the case of a force) suggests that they may be represented as arrows whose length is proportional to the magnitude. The other attributes like direction, orientation and the anchor point are more or less self-explanatory.

Geometrically, we identify a *vector* with an *ordered pair of points* *AB*, point *A* being the *anchor point* or the *origin* of the vector while the location of *B* depends on the remaining attributes of the quantity which is being represented by the vector. Usually we place an arrow above AB, $\overrightarrow{AB}$, to denote the vector with the origin *A* and the endpoint *B*.



Two vectors anchored at a point *A* can be added using the parallelogram rule. The sum is also anchored at *A*. A vector can be *scaled* by a number, a *scalar*. Scaling preserves the origin and the direction of the vector. It may affect the orientation (if the scalar is negative) and the length (if the scalar is different from both 1 and –1). Hence, in order to create the algebra of vectors we consider the set of vectors anchored at a single point.



In order to use algebraic approach to vectors we consider the space $\mathbb{R}^2$ or $\mathbb{R}^3$ or some such and we assume that all vectors originate at (0, … ,0). Thus, every vector is uniquely identified by a single point namely, its endpoint. This strategy results in a very easy algebraic definition of vector operations. If you have vectors v_1 and v_2 represented by their respective endpoints (a,b) and (c,d) then v_1+v_2 is represented by (a+c,b+d) and p(a,b) by (pa,pb)

We often write $(a,b) + (c,d) = (a+c,b+d)$ and $p(a,b) = (pa,pb)$ but you should be aware that this does not mean that we add or scale points of the plane (or other Euclidean space). We add and scale vectors who by default originate at (0,0) and terminate at (a,b) and (c,d) respectively.

Definition. (Vector space, formal definition)

A *vector space* (also called a *linear space*) is an ordered triple $(V, \mathbb{K}, f)$ where

- V is an Abelian group with the operation usually denoted by $+$, whose elements are called *vectors*,
- $\mathbb{K}$ is a field with operations denoted, somewhat confusingly by $+$ and by $\cdot$. Elements of $\mathbb{K}$ are called *scalars*,
- f is a function from $\mathbb{K} \times V$ into V called *scaling*. $f(p, v)$ is often, confusingly, denoted by $p \cdot v$,

such that

1. $(\forall \lambda \in \mathbb{K})(\forall u, v \in V)\ \lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$
2. $(\forall \alpha, \beta \in \mathbb{K})(\forall v \in V)\ (\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$
3. $(\forall \alpha, \beta \in \mathbb{K})(\forall v \in V)\ (\alpha \cdot \beta) \cdot v = \alpha \cdot (\beta \cdot v)$
4. $(\forall v \in V)\ 1 \cdot v = v$, where 1 denotes the identity element of the field multiplication (second operation).

Notice the ambiguity caused by the double meaning of the + symbol. This is a BAD, UGLY monster but it is traditional. We let the context decide which "+" means scalar-to-scalar and which vector-to-vector addition. Otherwise we would have to introduce extra symbols for scaling and vector addition that would also confuse people. And would be hard to type.

Similar remark applies to the dot $\cdot$, denoting both scaling (i.e. scalar-by-vector multiplication) and scalar-by-scalar multiplication.

Another problem is caused by the identity elements. In general statements about vector spaces some people use 0 to denote the identity element of both the scalar-to-scalar addition and vector-to-vector addition and let the context worry. Other people, though, myself included, make an effort to distinguish between the two using symbols like $\mathbf{0}$ (boldface zero), $\mathbb{0}$, θ or Θ to denote the "zero vector". The problem is that sometimes context is not enough, e.g. $0 \cdot 0$ makes sense both in case when the second 0 is the zero scalar and when the second 0 is the zero vector.

Examples.

Let $\mathbb{K}$ be any field and let $n \in \mathbb{N}$ be a natural number. Then

$\mathbb{K}^n = \{(x_1, x_2, \dots, x_n)\colon\ (\forall i = 1,2, \dots n)\ x_i \in \mathbb{K}\}$

together with vector addition defined as follows:

$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
(*called component-wise addition*)

and scaling $\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$ (*component-wise scaling*)

forms a vector space over $\mathbb{K}$.

The zero vector is $\Theta = (0,0, \dots, 0)$, the inverse of a vector $v = (a_1, a_2, \dots a_n)$ is $(-a_1, -a_2, \dots, -a_n)$.

Notice that in case of $n = 1$ we get (more or less) that every field is a vector space over itself.

Examples.

Let $\mathbb{K}$ be a field. The set $\mathbb{K}[x]$ of all polynomials over $\mathbb{K}$ with the standard polynomial addition and multiplication by a constant from $\mathbb{K}$ forms a vector space over $\mathbb{K}$. The zero vector 0 is the zero polynomial $\mathbf{0}$. Similarly, the set $\mathbb{K}_n[x]$ of polynomials of degree less than or equal to n over the field $\mathbb{K}$ is a vector space.

Example (generalized version of the previous one)

Let $\mathbb{K}$ be a field and let X be a set (any set). Let $V = \mathbb{K}^X$. We define function addition and scaling as the usual operations on functions (i.e. $(f+g)(x) = f(x) + g(x)$, where the second plus denotes operation one in $\mathbb{K}$ and $(\alpha \cdot f)(x) = \alpha \cdot f(x)$). $\mathbb{K}^X$ is a vector space over $\mathbb{K}$.

Proof. V is obviously closed under + (the sum of two functions from V exists and is a function from V). Is $f + (g + h) = (f + g) + h$? WTH does it mean that two functions are equal? They have the same domain and range, which is obvious, and ($\forall x \in$ X) $[f + (g + h)](x) = [(f + g) + h](x)$. The LHS = $f(x) + (g + h)(x) = f(x) + \big(g(x) + h(x)\big) = \big(f(x) + g(x)\big) + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x)$ = RHS.

In the same way we can show that + is commutative. What, if anything, is the zero vector, Θ? We define $\Theta(x) = 0$ for every $x \in X$, where "0" denotes the zero scalar.

Proof. (continued)

The inverse for f (w.r.t. +) is $(-f)$ defined as $(-f)(x) = -(f(x))$, where the second minus denotes the inverse in $\mathbb{K}$ of an element of $\mathbb{K}$ w.r.t. +.

Remaining axioms can be verified in the similar fashion. In each case the identity to be verified boils down to an axiom of fields. E.g.,

scaling

$$(\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f$$

scalar addition vector addition

follows from

field multiplication

$$(\alpha + \beta) \cdot f(x) = \alpha \cdot f(x) + \beta \cdot f(x) \quad \text{for all } x \in X.$$

field addition

(distributivity law in the field)

Example. (A REALLY outlandish one)

Let X be any set. We will use $V = (2^X, \div)$ as the Abelian group of vectors, where ÷ denotes the operation of symmetric difference of sets, A÷B = $(A \cup B) \setminus (A \cap B)$. We will also use $(\mathbb{Z}_2, \oplus, \otimes)$ as the field of scalars. Scaling is defined as follows:

for every set A, 0·A = ∅ and 1·A = A.

Comprehension.
Check that $(V, \mathbb{Z}_2, \cdot)$ is a vector space.

FAQ. 1

What the hell is a vector?
The only proper answer to this question, even though a little confusing, is *"A vector is an element of a vector space"*. The previous example teaches us that sets can be vectors. In other examples we have seen numbers, complex numbers, n-tuples of numbers, functions, polynomials etc. playing the role of vectors.

FAQ. 2
What the hell is a scalar then?

Well, you probably realize that the answer will be equally trivial (or disturbing). Any element of a field $\mathbb{K}$ may be called a scalar if somebody decides to construct a vector space using $\mathbb{K}$ as the second element of the ordered triple constituting a vector space. In particular, if we consider $\mathbb{K}$ a vector space over itself then all elements of $\mathbb{K}$ are at the same time scalars and vectors.

Example.
In the vector space of real numbers over the field of real numbers, real numbers are both vectors and scalars.

In $\mathbb{C}$ over $\mathbb{R}$ complex numbers are vectors, real numbers are scalars.

In 2^X over $\mathbb{Z}_2$vectors are subsets of X and there are but two scalars, 0 and 1.

What makes general study of vector spaces useful is that whatever facts we discover about vector spaces in general they are true in each of these spaces.

Theorem. (Arithmetic properties of vector spaces)
In every vector space V over a field $\mathbb{K}$

1. for every vector v, $0 \cdot v = \Theta$, (Θ stands for the zero vector).
2. for every scalar p, $p \cdot \Theta = \Theta$.
3. for every scalar p and for every vector v, $(-p) \cdot v = p \cdot (-v) = -(p \cdot v)$.
4. for every scalar p and for every vector v, $p \cdot v = \Theta$ implies $p = 0$ or $v = \Theta$.

Proof (left as a comprehension exercise).